DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **36**, Number **12**, December **2016** pp. **X–XX**

CLASSIFICATION OF POSITIVE SOLUTIONS TO A LANE-EMDEN TYPE INTEGRAL SYSTEM WITH NEGATIVE EXPONENTS

Jingbo Dou

School of Statistics
Xi'an University of Finance and Economics
Xi'an, Shaanxi 710100, China

Fangfang Ren

School of National Fiscal Development
Central University of Finance and Economics
Beijing 100081, China
and
School of Mathematics and Computational Science
Sun Yat-sen University, Guangzhou
Guangzhou 510275, China

John Villavert *

School of Mathematical and Statistical Sciences
University of Texas Rio Grande Valley
Edinburg, TX 78539, USA

Abstract. In this paper, we classify the positive solutions to the following Lane-Emden type integral system with negative exponents

$$\begin{cases} u(x) &= \displaystyle\int_{\mathbb{R}^n} |x-y|^{\tau} u^{-p}(y) v^{-q}(y)\, dy,\ x \in \mathbb{R}^n, \\ v(x) &= \displaystyle\int_{\mathbb{R}^n} |x-y|^{\tau} u^{-r}(y) v^{-s}(y)\, dy,\ x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 1$ is an integer and $\tau, p, q, r, s > 0$. Particularly, using an integral form of the method of moving spheres, we classify the positive solutions to the integral system whenever

$$p + q = r + s = 1 + 2n/\tau.$$

We also establish the non-existence of positive solutions under the condition

$$\max\{p+q, r+s\} \leq 1 + 2n/\tau \ \text{ and } \ p+q+r+s < 2(1+2n/\tau).$$

2010 *Mathematics Subject Classification.* Primary: 45G15; Secondary: 35B65, 35B53.

Key words and phrases. Integral equations, Lane-Emden system, negative exponent, regularity, method of moving spheres.

The first author is supported by the National Natural Science Foundation of China (Grant No. 11571268) and Natural Science Basic Research Plan in Shaanxi Province of China (Grant No. 2014JM1021).

* Corresponding author: John Villavert.

1. **Introduction.** In this article, we examine the regularity, classification and nonexistence of positive solutions to the following system of integral equations having coupled nonlinearities with negative exponents:

$$\begin{cases} u(x) &= \displaystyle\int_{\mathbb{R}^n} |x-y|^{\tau} u^{-p}(y) v^{-q}(y)\,dy, \ x \in \mathbb{R}^n, \\ v(x) &= \displaystyle\int_{\mathbb{R}^n} |x-y|^{\tau} u^{-r}(y) v^{-s}(y)\,dy, \ x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $n \geq 1$ is an integer and $\tau, p, q, r, s > 0$. Our motivation for studying this integral system stems from the fact that it arises naturally in the study of reversed variants of the Hardy-Littlewood-Sobolev (HLS) inequalities and in curvature problems from conformal geometry. For instance, a special case of system (1) is the integral system

$$\begin{cases} u(x) &= \displaystyle\int_{\mathbb{R}^n} |x-y|^{\tau} v^{-p}(y)\,dy, \ x \in \mathbb{R}^n, \\ v(x) &= \displaystyle\int_{\mathbb{R}^n} |x-y|^{\tau} u^{-q}(y)\,dy, \ x \in \mathbb{R}^n, \end{cases} \tag{2}$$

which is closely related to the Euler-Lagrange equation for the extremals to a reversed HLS inequality introduced by the first author and Zhu [8] (see also [22]). In particular, for $\tau = \alpha - n > 0$ and $p = q = -(n+\alpha)/(n-\alpha)$, the authors employed the method of moving spheres to show that every positive measurable solution of system (2) has the form

$$\begin{cases} u(x) &= a_1\big(|x-x_0|^2 + d\big)^{\frac{\alpha-n}{2}}, \ x \in \mathbb{R}^n, \\ v(x) &= a_2\big(|x-x_0|^2 + d\big)^{\frac{\alpha-n}{2}}, \ x \in \mathbb{R}^n, \end{cases} \tag{3}$$

where $x_0 \in \mathbb{R}^n$ is some point and $a_1, a_2, d > 0$ are constants. This classification result is a crucial step in finding the best constant in the reversed HLS inequality. For more on HLS inequalities and its reversed versions on, say, compact Riemannian manifolds and their applications to curvature problems, we refer the reader to [7, 10] and the references therein. Soon after, the author in [12] considered system (2) and obtained necessary conditions for the existence of positive solutions as well as necessary and sufficient conditions for the scale invariance of the system with respect to certain energy functionals.

When $u \equiv v$ and $p = q$, system (2) becomes the single integral equation

$$u(x) = \int_{\mathbb{R}^n} |x-y|^{\tau} u^{-p}(y)\,dy, \ x \in \mathbb{R}^n, \tag{4}$$

which was introduced by Li in [18]. Interestingly, when $n = 3$, $\tau = 1$ and $p = 7$, this equation is closely related to a fourth order conformal covariant operator on compact 3-manifolds. Xu [24] later proved that equation (4) has a positive solution of class C^1 if and only if $p = 1+2n/\tau$. Therefore the earlier result of Li [18] indicates that $p = 1 + 2n/\tau$ and the positive solution u has the form

$$u(x) = a(|x-x_0|^2 + d)^{\tau/2}, \ x \in \mathbb{R}^n, \tag{5}$$

where $x_0 \in \mathbb{R}^n$ is some point and $a, d > 0$ are constants. Although there are some similarities with the classical HLS integral equations, the results for equation (4) are somewhat surprising. More precisely, if $\alpha \in (0, n)$ and $p > 0$, Chen, Li and Ou in [2] and [4] proved the following for the classical HLS integral equation

$$u(x)=\int_{\mathbb{R}^n}|x-y|^{\alpha-n}u^p(y)\,dy,\ x\in\mathbb{R}^n. \tag{6}$$

(a) Every positive regular solution of (6) in the critical case

$$p=(n+\alpha)/(n-\alpha)$$

has the form (5) but with $\tau=-(n-\alpha)<0$.

(b) The only non-negative regular solution of (6) is $u\equiv 0$ whenever the subcritical condition holds, i.e.,

$$p<(n+\alpha)/(n-\alpha).$$

Unlike with equation (4), however, there does exist positive solutions in the supercritical case $p>(n+\alpha)/(n-\alpha)$, at least when α is an even integer (see [14, 15, 20]). Furthermore, analogous results—albeit mostly partial ones—are known for the HLS system, i.e., when $p,q<0$ and $\tau=\alpha-n<0$ in (2). Namely, the questions on the classification, existence and non-existence of positive solutions remain open for the most part. We refer the reader to the papers [1, 3, 5, 11, 13, 19] and the references therein for more details.

If $p>1$ and $\alpha\in(0,n)$, it is noteworthy to mention the equivalence between equation (6) and the partial differential equation

$$(-\Delta)^{\alpha/2}u(x)=u^p(x),\ u>0,\ x\in\mathbb{R}^n.$$

Here we mean the two equations are equivalent if, assuming solutions belong to the appropriate function space, a positive solution of one equation multiplied by a suitable positive constant if necessary, is also a positive solution of the other; and vice versa (cf. [2, 23]). Therefore, the results for the integral equation also hold for the equivalent differential equation, and this illustrates one advantage of studying the integral equations. In view of this, one can obviously consider the corresponding differential equations to system (1). Indeed, several papers have addressed the regularity, existence and non-existence of positive solutions to such differential systems with negative exponents on bounded smooth domains (see [9, 25]). We should also mention several past works that examine system (1) but with $\tau=\alpha-n<0$, $p=s\leq-1$, $q=r\leq-1$ and its corresponding differential system (sometimes called the Schrödinger type elliptic system). For example, Li and Ma [21] studied the symmetry and uniqueness of its positive ground state solutions. Inspired by this, the first author of this paper examined the same integral system and further obtained classification results when $p+q=-(n+\alpha)/(n-\alpha)$ and non-existence results when $p+q>-(n+\alpha)/(n-\alpha)$ (see [6]). Using a topological approach, Li and the third author [14] recently obtained existence results for a family of elliptic systems which included the Schrödinger type system.

Motivated by the previous results for the above integral equations and systems, our aim in this paper is to generate similar classification and non-existence results for system (1). We achieve this by utilizing similar tools developed in the earlier works described above. In particular, we exploit an integral version of the method of moving spheres (see [16, 17, 18]). In the process, however, we must address and overcome several issues contributed by the coupled components and negative exponents in the problem.

We now state our main results, which are reminiscent of the ones for equation (4). We begin with a theorem on the regularity of measurable solutions. In this paper, measurable solutions refer to solutions which are Lebesgue measurable and non-infinity.

Theorem 1.1. *Let $n \geq 1$, $\tau, p, q, r, s > 0$ and let (u, v) be a pair of positive measurable solutions to system (1). Then u, v are smooth, i.e., u, v belong to $C^\infty(\mathbb{R}^n)$.*

This theorem indicates that we can always assume hereafter that solutions of system (1) are smooth. Then the following classification result holds for positive solutions.

Theorem 1.2. *Suppose $n \geq 1$ and $\tau, p, q, r, s > 0$ satisfy*

$$p + q = r + s = 1 + 2n/\tau.$$

If (u, v) is a pair of positive smooth solutions to system (1), then u, v have the form

$$\begin{cases} u(x) & = c_1\big(|x - x_0|^2 + d\big)^{\tau/2}, \ x \in \mathbb{R}^n, \\ v(x) & = c_2\big(|x - x_0|^2 + d\big)^{\tau/2}, \ x \in \mathbb{R}^n, \end{cases} \tag{7}$$

where $x_0 \in \mathbb{R}^n$ is some point and $c_1, c_2, d > 0$ are constants.

We now address the non-existence of solutions for the integral system. Although the next lemma plays an important role in our proofs of Theorems 1.1 and 1.2, we state it here because it also yields a non-existence result. Its proof is straightforward and so we state and prove it right after. This is not so much the case for the lemma itself, so we delay its proof until the next section.

Lemma 1.3. *For $n \geq 1$ and $\tau, p, q, r, s > 0$, if (u, v) is a pair of positive measurable solutions to (1), then*

$$\begin{aligned} (i) \qquad & \int_{\mathbb{R}^n} (1 + |y|^\tau) u^{-p}(y) v^{-q}(y)\, dy < \infty, \\ & \int_{\mathbb{R}^n} (1 + |y|^\tau) u^{-r}(y) v^{-s}(y)\, dy < \infty; \\ (ii) \qquad & a := \lim_{|x|\to\infty} |x|^{-\tau} u(x) = \int_{\mathbb{R}^n} u^{-p}(y) v^{-q}(y)\, dy < \infty, \\ & b := \lim_{|x|\to\infty} |x|^{\tau} v(x) = \int_{\mathbb{R}^n} u^{-r}(y) v^{-s}(y)\, dy < \infty; \\ (iii) \qquad & \textit{For some constants } C_1, C_2 > 0, \\ & \frac{1 + |x|^\tau}{C_1} \leq u(x) \leq C_1(1 + |x|^\tau), \ \forall\, x \in \mathbb{R}^n, \\ & \frac{1 + |x|^\tau}{C_2} \leq v(x) \leq C_2(1 + |x|^\tau), \ \forall\, x \in \mathbb{R}^n. \end{aligned}$$

As noted above, we can easily deduce a non-existence result from Lemma 1.3. To see this, let (u, v) be a pair of positive measurable solutions to system (1). Without loss of generality, we can assume that

$$p + q = \max\{p + q, r + s\}.$$

Then $p + q > 1 + n/\tau$ is clearly a necessary condition for the existence of positive solutions. On the contrary, i.e., if $n + \tau - \tau(p + q) \geq 0$, Lemma 1.3 (iii) would then imply that

$$u(x) \geq \int_{\mathbb{R}^n \setminus B_1(0)} |x - y|^\tau u^{-p}(y) v^{-q}(y)\, dy \geq C \int_1^\infty r^{n+\tau-\tau(p+q)} \, \frac{dr}{r} = \infty$$

for a.e. $x \in \mathbb{R}^n$. Essentially, we have proved that

Theorem 1.4. *System* (1) *admits no positive smooth solution whenever*

$$\max\{p+q,\, r+s\} \le 1 + n/\tau.$$

Of course, one may ask if this non-existence result is optimal. It turns out that this is not the case, and we adapt the method from our proof of Theorem 1.2 to get an improved version.

Theorem 1.5. *Let $n \ge 1$ and $\tau, p, q, r, s > 0$ satisfy*

$$\max\{p+q, r+s\} \le 1 + 2n/\tau \text{ and } p+q+r+s < 2\big(1+2n/\tau\big).$$

Then system (1) *admits no positive smooth solution.*

The remaining parts of this paper is arranged in the following manner. In Section 2, we provide the proof of Lemma 1.3 followed by the proof of Theorem 1.1. Section 3 contains the proof of Theorem 1.2 and Section 4 contains the proof of Theorem 1.5.

2. **Regularity.** In this section, we establish the regularity of positive solutions to (1), but first we give the proof of Lemma 1.3. Throughout the paper, $B_R(x_0)$ denotes the set $\{x \in \mathbb{R}^n : |x - x_0| < R\}$, the open ball of radius $R > 0$ with center $x_0 \in \mathbb{R}^n$. We sometimes use the short-hand notation $B_R = B_R(0)$.

Proof of Lemma 1.3. The proof is similar to that of Lemma 5.1 in Li [18], but we include it here for completeness. Since u and v are non-infinity measurable functions, we have

$$meas\{x \in \mathbb{R}^n : u(x) < \infty\} > 0, \quad \text{and} \quad meas\{x \in \mathbb{R}^n : v(x) < \infty\} > 0.$$

Moreover, there exist $R > 1$ and some measurable set E such that

$$E \subset \{x \in \mathbb{R}^n : u(x), v(x) < R\} \cap B_R$$

with $|E| > \frac{1}{R}$. For any $x \in \mathbb{R}^n$, there holds

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^n} |x-y|^\tau u^{-p}(y) v^{-q}(y) dy \\ &\ge \int_E |x-y|^\tau u^{-p}(y) v^{-q}(y) dy \\ &\ge R^{-(p+q)} \int_E |x-y|^\tau dy. \end{aligned}$$

Then

$$\lim_{|x|\to\infty} \frac{u(x)}{(1+|x|^\tau)} \ge \lim_{|x|\to\infty} \frac{R^{-(p+q)}}{(1+|x|^\tau)} \int_E |x-y|^\tau dx = CR^{-(p+q)-1},$$

which implies

$$u(x) \ge \frac{(1+|x|^\tau)}{C_1}.$$

Similarly, for any $x \in \mathbb{R}^n$, we have

$$v(x) \ge \frac{(1+|x|^\tau)}{C_2}.$$

This proves the left hand side of the inequalities in (iii).

On the other hand, for some $x_0 \in \mathbb{R}^n$ with $1 \le |x_0| \le 2$,

$$\int_{\mathbb{R}^n} |x_0 - y|^\tau u^{-p}(y) v^{-q}(y) dy = u(x_0) < \infty,$$

$$\int_{\mathbb{R}^n} |x_0 - y|^\tau u^{-r}(y) v^{-s}(y) dy = v(x_0) < \infty.$$

Combining the left hand side inequalities in (iii) and the above, we get (i).

For $|x| \ge 1$,

$$\frac{|x-y|^\tau}{|x|^\tau} u^{-p}(y) v^{-q}(y) \le (1 + |y|^\tau) u^{-p}(y) v^{-q}(y),$$

and

$$\frac{|x-y|^\tau}{|x|^\tau} u^{-r}(y) v^{-s}(y) \le (1 + |y|^\tau) u^{-r}(y) v^{-s}(y).$$

Taking these with (i) and using the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} a &= \lim_{|x|\to\infty} |x|^{-\tau} u(x) = \lim_{|x|\to\infty} \int_{\mathbb{R}^n} \frac{|x-y|^\tau}{|x|^\tau} u^{-p}(y) v^{-q}(y)\, dy \\ &= \int_{\mathbb{R}^n} u^{-p}(y) v^{-q}(y)\, dy < \infty, \\ \textit{and} & \\ b &= \lim_{|x|\to\infty} |x|^{-\tau} v(x) = \lim_{|x|\to\infty} \int_{\mathbb{R}^n} \frac{|x-y|^\tau}{|x|^\tau} u^{-r}(y) v^{-s}(y)\, dy \\ &= \int_{\mathbb{R}^n} u^{-r}(y) v^{-s}(y)\, dy < \infty. \end{aligned}$$

We obtain (ii). Combining (i) and (ii) with (1), we get the right-hand side of the inequality in (iii). □

Proof of Theorem 1.1. For an arbitrary choice of $R > 0$, we can split u into two parts:

$$\begin{aligned} u(x) &= \int_{|y|\le 2R} |x-y|^\tau u^{-p}(y) v^{-q}(y)\, dy + \int_{|y|>2R} |x-y|^\tau u^{-p}(y) v^{-q}(y)\, dy \\ &=: J_1(x) + J_2(x). \end{aligned}$$

Applying Lemma 1.3 (i), $J_2(x)$ can be differentiated under the integral for $|x| < R$, so $J_2 \in C^\infty(B_R)$. On the other hand, by Lemma 1.3 (iii), we have $u^{-p}v^{-q} \in L^\infty(B_{2R})$ and so J_1 is at least Hölder continuous in B_R. Since $R > 0$ is arbitrary, u is at least Hölder continuous in $\mathbb{R}^n$, and along a similar process, we can deduce that v is at least Hölder continuous in $\mathbb{R}^n$. So in view of Lemma 1.3 (iii), we have that $u^{-p}v^{-q}$ is Hölder continuous in B_{2R} and the regularity of J_1 is further improved. By standard bootstrap arguments, we conclude that $u \in C^\infty(\mathbb{R}^n)$. Likewise, a similar argument shows that $v \in C^\infty(\mathbb{R}^n)$. This completes the proof of the theorem. □

3. **Classification of positive solutions in the critical case.** In this section, we complete the proof of Theorem 1.2. To this end, we employ the Kelvin transform and the method of moving spheres of Li and Zhu [16], which was later improved by Li [18] (see also Dou and Zhu [8]).

For $x \in \mathbb{R}^n$ and $\lambda > 0$, we define

$$\omega_{x,\lambda}(\xi) = \big(\frac{\lambda}{|\xi - x|}\big)^{-\tau} \omega(\xi^{x,\lambda}), \quad \forall \xi \in \mathbb{R}^n \setminus \{x\},$$

where

$$\xi^{x,\lambda} = x + \frac{\lambda^2(\xi - x)}{|\xi - x|^2}$$

is the Kelvin transform of ξ with respect to $B_\lambda(x)$. Set $\Sigma_{x,\lambda} = \mathbb{R}^n \backslash \overline{B_\lambda(x)}$.

Lemma 3.1. *Let $\tau > 0$ and $p, q, r, s > 0$. If (u, v) is a pair of positive solutions to system (1), then, for any $x \in \mathbb{R}^n$,*

$$u_{x,\lambda}(\xi) = \int_{\mathbb{R}^n} |\xi - \eta|^\tau u_{x,\lambda}^{-p}(\eta) v_{x,\lambda}^{-q}(\eta) \big(\frac{\lambda}{|\eta - x|}\big)^{\theta_1} d\eta, \quad \forall\, \xi \in \mathbb{R}^n, \tag{8}$$

$$v_{x,\lambda}(\xi) = \int_{\mathbb{R}^n} |\xi - \eta|^\tau u_{x,\lambda}^{-r}(\eta) v_{x,\lambda}^{-s}(\eta) \big(\frac{\lambda}{|\eta - x|}\big)^{\theta_2} d\eta, \quad \forall\, \xi \in \mathbb{R}^n, \tag{9}$$

where

$$\theta_1 = 2n + \tau(1 - p - q), \theta_2 = 2n + \tau(1 - r - s).$$

Moreover,

$$u_{x,\lambda}(\xi) - u(\xi) = \int_{\Sigma_{x,\lambda}} K(x, \lambda; \xi, \eta) \big[u^{-p}(\eta) v^{-q}(\eta) - \big(\frac{\lambda}{|\eta - x|}\big)^{\theta_1} u_{x,\lambda}^{-p}(\eta) v_{x,\lambda}^{-q}(\eta)\big] d\eta, \tag{10}$$

$$v_{x,\lambda}(\xi) - v(\xi) = \int_{\Sigma_{x,\lambda}} K(x, \lambda; \xi, \eta) \big[u^{-r}(\eta) v^{-s}(\eta) - \big(\frac{\lambda}{|\eta - x|}\big)^{\theta_2} u_{x,\lambda}^{-r}(\eta) v_{x,\lambda}^{-s}(\eta)\big] d\eta, \tag{11}$$

where

$$K(x, \lambda; \xi, \eta) = \big(\frac{\lambda}{|\xi - x|}\big)^{-\tau} |\xi^{x,\lambda} - \eta|^\tau - |\xi - \eta|^\tau,$$

and

$$K(x, \lambda; \xi, \eta) > 0, \ \forall\, \xi, \eta \in \Sigma_{x,\lambda}, \lambda > 0.$$

Proof. The lemma can be verified via direct calculations, but we sketch the proof for the reader's convenience. Write

$$y = \eta^{x,\lambda} = x + \frac{\lambda^2(\eta - x)}{|\eta - x|^2}$$

with $x, \eta \in \mathbb{R}^n$ and $\lambda > 0$. The n-space forms in the y and η variables are related by

$$dy = \big(\frac{\lambda}{|\eta - x|}\big)^{2n} d\eta.$$

For simplicity, write

$$A^+(\xi^{x,\lambda}) = \int_{\Sigma_{x,\lambda}} |\xi^{x,\lambda} - y|^\tau u^{-p}(y) v^{-q}(y)\, dy,$$

$$A^-(\xi^{x,\lambda}) = \int_{B_\lambda(x)} |\xi^{x,\lambda} - y|^\tau u^{-p}(y) v^{-q}(y)\, dy.$$

From (1), u can then be rewritten as

$$u(\xi^{x,\lambda}) = A^+(\xi^{x,\lambda}) + A^-(\xi^{x,\lambda})$$

for $x, \xi \in \mathbb{R}^n$. By the change of variables we have

$$\begin{aligned} A^+(\xi^{x,\lambda}) &= \int_{\Sigma_{x,\lambda}} |\xi^{x,\lambda} - y|^\tau u^{-p}(y) v^{-q}(y)\, dy \\ &= \int_{B_\lambda(x)} |\xi^{x,\lambda} - \eta^{x,\lambda}|^\tau u^{-p}(\eta^{x,\lambda}) v^{-q}(\eta^{x,\lambda}) \big(\frac{\lambda}{|\eta - x|}\big)^{2n} d\eta \\ &= \int_{B_\lambda(x)} |\xi^{x,\lambda} - \eta^{x,\lambda}|^\tau u_{x,\lambda}^{-p}(\eta) v_{x,\lambda}^{-q}(\eta) \big(\frac{\lambda}{|\eta - x|}\big)^{2n-\tau(p+q)}\, d\eta. \end{aligned}$$

Using the fact that

$$\frac{|\eta - x|}{\lambda} \frac{|\xi - x|}{\lambda} |\xi^{x,\lambda} - \eta^{x,\lambda}| = |\xi - \eta|,$$

we arrive at

$$\begin{aligned} A_{x,\lambda}^+(\xi) &:= \big(\frac{\lambda}{|\xi - x|}\big)^{-\tau} A^+(\xi^{x,\lambda}) \\ &= \big(\frac{\lambda}{|\xi - x|}\big)^{-\tau} \int_{B_\lambda(x)} |\xi^{x,\lambda} - \eta^{x,\lambda}|^\tau u_{x,\lambda}^{-p}(\eta) v_{x,\lambda}^{-q}(\eta) \big(\frac{\lambda}{|\eta - x|}\big)^{2n-\tau(p+q)} d\eta \\ &= \int_{B_\lambda(x)} |\xi - \eta|^\tau u_{x,\lambda}^{-p}(\eta) v_{x,\lambda}^{-q}(\eta) \big(\frac{\lambda}{|\eta - x|}\big)^{\theta_1} d\eta. \end{aligned}$$

Similarly, we have

$$A_{x,\lambda}^-(\xi) := \big(\frac{\lambda}{|\xi - x|}\big)^{-\tau} A^-(\xi^{x,\lambda}) = \int_{\Sigma_{x,\lambda}} |\xi - \eta|^\tau u_{x,\lambda}^{-p}(\eta) v_{x,\lambda}^{-q}(\eta) \big(\frac{\lambda}{|\eta - x|}\big)^{\theta_1} d\eta.$$

Hence,

$$u_{x,\lambda}(\xi) = A_{x,\lambda}^+(\xi) + A_{x,\lambda}^-(\xi) = \int_{\mathbb{R}^n} |\xi - \eta|^\tau u_{x,\lambda}^{-p}(\eta) v_{x,\lambda}^{-q}(\eta) \big(\frac{\lambda}{|\eta - x|}\big)^{\theta_1} d\eta.$$

Identity (8) is established. Furthermore,

$$\begin{aligned} u_{x,\lambda}(\xi) - u(\xi) &= A_{x,\lambda}^+(\xi) + A_{x,\lambda}^-(\xi) - \big(A^+(\xi) + A^-(\xi)\big) \\ &= (A_{x,\lambda}^-(\xi) - A^+(\xi)) + (A_{x,\lambda}^+(\xi) - A^-(\xi)) \\ &= \int_{\Sigma_{x,\lambda}} |\xi - \eta|^\tau \big[\big(\frac{\lambda}{|\eta - x|}\big)^{\theta_1} u_{x,\lambda}^{-p}(\eta) v_{x,\lambda}^{-q}(\eta) - u^{-p}(\eta) v^{-q}(\eta)\big] d\eta \\ &\quad + (A_{x,\lambda}^+(\xi) - A^-(\xi)). \end{aligned}$$

Noting that

$$\begin{aligned} A_{x,\lambda}^+(\xi) &= \big(\frac{\lambda}{|\xi - x|}\big)^{-\tau} A^+(\xi^{x,\lambda}) \\ &= \big(\frac{\lambda}{|\xi - x|}\big)^{-\tau} \int_{\Sigma_{x,\lambda}} |\xi^{x,\lambda} - y|^\tau u^{-p}(y) v^{-q}(y)\, dy, \end{aligned}$$

and using the fact $(\xi^{x,\lambda})^{x,\lambda} = \xi$, we have

$$\begin{aligned} A^-(\xi) &= A^-((\xi^{x,\lambda})^{x,\lambda}) = \big(\frac{\lambda}{|\xi^{x,\lambda} - x|}\big)^\tau A_{x,\lambda}^-(\xi^{x,\lambda}) \\ &= \big(\frac{\lambda}{|\xi^{x,\lambda} - x|}\big)^\tau \int_{\Sigma_{x,\lambda}} |\xi^{x,\lambda} - \eta|^\tau u_{x,\lambda}^{-p}(\eta) v_{x,\lambda}^{-q}(\eta) \big(\frac{\lambda}{|\eta - x|}\big)^{\theta_1} d\eta \\ &= \big(\frac{\lambda}{|\xi - x|}\big)^{-\tau} \int_{\Sigma_{x,\lambda}} |\xi^{x,\lambda} - \eta|^\tau u_{x,\lambda}^{-p}(\eta) v_{x,\lambda}^{-q}(\eta) \big(\frac{\lambda}{|\eta - x|}\big)^{\theta_1} d\eta. \end{aligned}$$

Identity (10) is established. Along the same line, we can deduce (9) and (11).

Finally, we discuss the sign of $K(x,\lambda;\xi,\eta)$. For $\xi,\eta\in\Sigma_{x,\lambda}$ and $\lambda>0$, it is easy to verify that

$$\begin{aligned}|\xi-\eta|^2-\big(\frac{|\xi-x|}{\lambda}\big)^2|\xi^{x,\lambda}-\eta|^2&=\big(|\xi-x|^2+2\langle\xi-x,x-\eta\rangle+|x-\eta|^2\big)\\&\quad-\big(\frac{|\xi-x|}{\lambda}\big)^2\big(|x-\eta|^2+\frac{2\lambda^2\langle x-\eta,\xi-x\rangle}{|\xi-x|^2}+\frac{\lambda^4}{|\xi-x|^2}\big)\\&=|\xi-x|^2+|x-\eta|^2-\frac{|\xi-x|^2|x-\eta|^2}{\lambda^2}-\lambda^2\\&=\big(\frac{|\xi-x|^2}{\lambda^2}-1\big)\big(\lambda^2-|x-\eta|^2\big)<0.\end{aligned}$$

Since $h(x)=x^{\tau/2}$ is montonically increasing on $(0,\infty)$, we have

$$\big(\frac{|\xi-x|}{\lambda}\big)^\tau|\xi^{x,\lambda}-\eta|^\tau-|\xi-\eta|^\tau=\big[\big(\frac{|\xi-x|}{\lambda}|\xi^{x,\lambda}-\eta|\big)^2\big]^{\frac{\tau}{2}}-\big(|\xi-\eta|^2\big)^{\frac{\tau}{2}}>0.$$

Hence, we conclude that $K(x,\lambda;\xi,\eta)>0$ for $\xi,\eta\in\Sigma_{x,\lambda}$ and $\lambda>0$. This completes the proof of Lemma 3.1. □

In Lemma 3.1, we note that $\theta_1=\theta_2=0$ if and only if $p+q=r+s=1+2n/\tau$. So from now on in this section, we assume that $p+q=r+s=1+2n/\tau$. Next, we prove Theorem 1.2 by the method of moving spheres, but we require some preliminary lemmas. The first lemma guarantees we can initiate the method of moving spheres.

Lemma 3.2. *Assume the same conditions as those in Theorem 1.2. Then there exists $\lambda_0(x)>0$ for each $x\in\mathbb{R}^n$ such that*

$$u_{x,\lambda}(\xi)\ge u(\xi),\ \textit{and}\ v_{x,\lambda}(\xi)\ge v(\xi),\quad\forall\ \xi\in\Sigma_{x,\lambda},\forall\ 0<\lambda<\lambda_0(x).$$

Proof. Without loss of generality, we may assume $x=0$ and write $u_\lambda=u_{0,\lambda}$. Since $\tau>0$ and $u\in C^1(\mathbb{R}^n)$ is a positive function, there exists $r_0\in(0,1)$ such that

$$\nabla_\xi\big(|\xi|^{-\frac{\tau}{2}}u(\xi)\big)\cdot\xi<0,\ \forall\,0<|\xi|<r_0.$$

Thus,

$$u_\lambda(\xi)>u(\xi),\ \forall\,0<\lambda<|\xi|<r_0.\tag{12}$$

From Lemma 1.3 (iii), we get

$$u(\xi)\le C(r_0)|\xi|^\tau,\ \forall\ |\xi|\ge r_0.$$

For small $\lambda_0\in(0,r_0)$ and any $0<\lambda<\lambda_0$, using (iii) of Lemma 1.3 and (12)

$$u_\lambda(\xi)=\big(\frac{\lambda}{|\xi|}\big)^{-\tau}u\big(\frac{\lambda^2\xi}{|\xi|^2}\big)\ge\big(\frac{|\xi|}{\lambda_0}\big)^\tau\inf_{B_{r_0}}u\ge u(\xi),\qquad|\xi|\ge r_0.$$

Combining the above with (12), we arrive at

$$u_{x,\lambda}(\xi)\ge u(\xi),\qquad\forall\xi\in\Sigma_{x,\lambda},\ 0<\lambda<\lambda_0(x)$$

with $x=0$ and $\lambda_0(x)=\lambda_0$. Likewise, we can use similar arguments to arrive at

$$v_{x,\lambda}(\xi)\ge v(\xi),\quad\forall\xi\in\Sigma_{x,\lambda},\ 0<\lambda<\lambda_0(x)\text{ with }x=0.$$

This completes the proof of the lemma. □

For a given $x \in \mathbb{R}^n$, define

$$\bar{\lambda}(x) = sup\{\mu > 0 \,|\, u_{x,\lambda}(\xi) \geq u(\xi) \text{ and } v_{x,\lambda}(\xi) \geq v(\xi),\ \forall \lambda \in (0,\mu), \forall\, \xi \in \Sigma_{x,\lambda}\}.$$

The next lemma shows that solutions must have the conformal invariance property provided that the sphere stops.

Lemma 3.3. *For some $x_0 \in \mathbb{R}^n$, if $\bar{\lambda}(x_0) < \infty$, then*

$$u_{x_0,\bar{\lambda}(x_0)}(\xi) = u(\xi), \text{ and } v_{x_0,\bar{\lambda}(x_0)}(\xi) = v(\xi), \quad \forall \xi \in\ \mathbb{R}^n.$$

Proof. Without loss of generality, we may assume that $x_0 = 0$ and we write $\bar{\lambda} = \bar{\lambda}(0)$, $u_\lambda = u_{0,\lambda}$, $v_\lambda = v_{0,\lambda}$, $\xi^\lambda = \xi^{0,\lambda}$ and $\Sigma_\lambda = \Sigma_{0,\lambda}$. By the definition of $\bar{\lambda}$,

$$u_{\bar{\lambda}}(\xi) \geq u(\xi), \quad v_{\bar{\lambda}}(\xi) \geq v(\xi), \quad \forall\, |\xi| \geq \bar{\lambda}.$$

Since $\theta_1 = \theta_2 = 0$, we use (10) and (11) with $x = 0, \lambda = \bar{\lambda}$ and the positivity of the kernel to arrive at the following two cases:
(**a**) $u_{\bar{\lambda}}(\xi) = u(\xi)$ and $v_{\bar{\lambda}}(\xi) = v(\xi)$ for all $|\xi| \geq \bar{\lambda}$, or
(**b**) $u_{\bar{\lambda}}(\xi) > u(\xi)$ and $v_{\bar{\lambda}}(\xi) > v(\xi)$ for all $|\xi| > \bar{\lambda}$.

We claim that case (**b**) cannot happen. More precisely, if $u_{\bar{\lambda}}(\xi) > u(\xi)$ and $v_{\bar{\lambda}}(\xi) > v(\xi)$ for all $|\xi| > \bar{\lambda}$, then we will show that there is a suitably small $\varepsilon_* > 0$ such that, for any $\lambda \in (\bar{\lambda}, \bar{\lambda} + \varepsilon_*)$, $u_\lambda(\xi) \geq u(\xi)$ and $v_\lambda(\xi) \geq v(\xi)$ for any $|\xi| > \lambda$. This contradicts with the definition of $\bar{\lambda}$ and would complete the proof of the lemma. So now we prove the claim in two steps.

Step 1. We claim that there exists an $\varepsilon_1 \in (0, 1)$, such that for any $\varepsilon < \varepsilon_1$, $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon$, if $|\xi| \geq \bar{\lambda} + 1$, then

$$u_\lambda(\xi) - u(\xi) \geq \frac{\varepsilon_1}{2}|\xi|^\tau \quad \text{and} \quad v_\lambda(\xi) - v(\xi) \geq \frac{\varepsilon_1}{2}|\xi|^\tau.$$

From Lemma 3.1, we know that $K(0, \lambda, \xi, \eta) > 0$ for $\forall\, \xi, \eta \in \Sigma_\lambda$. So using (10) and Fatou's lemma, we know that for all $|\xi| \geq \bar{\lambda}$,

$$\begin{aligned}
&\liminf_{|\xi|\to\infty} |\xi|^{-\tau}(u_{\bar{\lambda}}(\xi) - u(\xi)) \\
&\quad \geq \int_{\Sigma_{\bar{\lambda}}} \liminf_{|\xi|\to\infty} |\xi|^{-\tau} K(0, \bar{\lambda}, \xi, \eta)\big[u^{-p}(\eta)v^{-q}(\eta) - u_{\bar{\lambda}}^{-p}(\eta)v_{\bar{\lambda}}^{-q}(\eta)\big]d\eta \\
&\quad = \int_{\Sigma_{\bar{\lambda}}} \big((\frac{\bar{\lambda}}{|\eta|})^{-\tau} - 1\big)\big[u^{-p}(\eta)v^{-q}(\eta) - u_{\bar{\lambda}}^{-p}(\eta)v_{\bar{\lambda}}^{-q}(\eta)\big]d\eta.
\end{aligned}$$

Due to the positivity of $u^{-p}(\eta)v^{-q}(\eta) - u_{\bar{\lambda}}^{-p}(\eta)v_{\bar{\lambda}}^{-q}(\eta)$, we get that there exists $\varepsilon_2 \in (0, 1)$ such that

$$u_{\bar{\lambda}}(\xi) - u(\xi) \geq \varepsilon_2|\xi|^\tau,\ \forall\, |\xi| \geq \bar{\lambda} + 1.$$

By the continuity of u, there exists an $\varepsilon_3 \in (0, \varepsilon_2)$ such that for $|\xi| \geq \bar{\lambda} + 1$ and $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon_3$,

$$\begin{aligned}
|u_\lambda(\xi) - u_{\bar{\lambda}}(\xi)| &= |\big(\frac{\lambda}{|\xi|}\big)^{-\tau} u(\frac{\lambda^2\xi}{|\xi|^2}) - \big(\frac{\bar{\lambda}}{|\xi|}\big)^{-\tau} u(\frac{\bar{\lambda}^2\xi}{|\xi|^2})| \\
&\leq \frac{\varepsilon_3}{2}|\xi|^\tau.
\end{aligned}$$

Thus, for all $|\xi| \geq \bar{\lambda} + 1, \bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon_2$,

$$u_\lambda(\xi) - u(\xi) = u_{\bar{\lambda}}(\xi) - u(\xi) + u_\lambda(\xi) - u_{\bar{\lambda}}(\xi) \geq \frac{\varepsilon_2}{2}|\xi|^\tau.$$

Similarly, there exists $\varepsilon_4 \in (0, \varepsilon_3)$ such that

$$v_\lambda(\xi) - v(\xi) \geq \frac{\varepsilon_4}{2} |\xi|^\tau,$$

for all $|\xi| \geq \bar{\lambda} + 1, \bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon_4$. Choosing $\varepsilon_1 = \varepsilon_4$, we complete the proof of the claim.

Step 2. There is an $\varepsilon_* < \varepsilon_1$, such that for any $\varepsilon < \varepsilon_*$, $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon$, if $\xi \in \mathbb{R}^n$ satisfies $\lambda \leq |\xi| \leq \bar{\lambda} + 1$, then $u_\lambda(\xi) - u(\xi) \geq 0$ and $v_\lambda(\xi) - v(\xi) \geq 0$.

Let $\varepsilon_* \in (0, \varepsilon_1)$. For $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon_*$ and $\lambda \leq |\xi| \leq \bar{\lambda} + 1$, we have,

$$\begin{aligned}
u_\lambda(\xi) - u(\xi) &= \int_{\Sigma_\lambda} K(0, \lambda; \xi, \eta) \big(u^{-p}(\eta) v^{-q}(\eta) - u_\lambda^{-p}(\eta) v_\lambda^{-q}(\eta)\big) d\eta \\
&\geq \int_{\Sigma_\lambda \setminus \Sigma_{\bar{\lambda}+1}} K(0, \lambda; \xi, \eta) \big(u^{-p}(\eta) v^{-q}(\eta) - u_\lambda^{-p}(\eta) v_\lambda^{-q}(\eta)\big) d\eta \\
&\quad + \int_{\Sigma_{\bar{\lambda}+2} \setminus \Sigma_{\bar{\lambda}+3}} K(0, \lambda; \xi, \eta) \big(u^{-p}(\eta) v^{-q}(\eta) - u_\lambda^{-p}(\eta) v_\lambda^{-q}(\eta)\big) d\eta \\
&\geq \int_{\Sigma_\lambda \setminus \Sigma_{\bar{\lambda}+1}} K(0, \lambda; \xi, \eta) \big(u_{\bar{\lambda}}^{-p}(\eta) v_{\bar{\lambda}}^{-q}(\eta) - u_\lambda^{-p}(\eta) v_\lambda^{-q}(\eta)\big) d\eta \\
&\quad + \int_{\Sigma_{\bar{\lambda}+2} \setminus \Sigma_{\bar{\lambda}+3}} K(0, \lambda; \xi, \eta) \big(u^{-p}(\eta) v^{-q}(\eta) - u_\lambda^{-p}(\eta) v_\lambda^{-q}(\eta)\big) d\eta.
\end{aligned} \tag{13}$$

By Step 1, there exists $\delta_1 > 0$ such that

$$u^{-p}(\eta) v^{-q}(\eta) - u_\lambda^{-p}(\eta) v_\lambda^{-q}(\eta) \geq \delta_1, \ \forall \eta \in \Sigma_{\bar{\lambda}+2} \setminus \Sigma_{\bar{\lambda}+3}.$$

Since

$$K(0, \lambda; \xi, \eta) = 0, \ \forall |\xi| = \lambda,$$

$$\nabla_\xi K(0, \lambda; \xi, \eta) \cdot \xi|_{|\xi|=\lambda} = \tau |\xi - \eta|^{\tau-2} \big(|\eta|^2 - |\xi|^2\big) > 0, \ \forall \eta \in \Sigma_{\bar{\lambda}+2} \setminus \Sigma_{\bar{\lambda}+3},$$

and the function is smooth in the relevant region, then, based on the positivity of kernel, we have

$$K(0, \lambda; \xi, \eta) \geq \delta_2(|\xi| - \lambda), \ \forall \lambda \leq |\xi| \leq \bar{\lambda} + 1, \forall \eta \in \Sigma_{\bar{\lambda}+2} \setminus \Sigma_{\bar{\lambda}+3},$$

where $\delta_2 > 0$ is some constant independent of ε_*. It is easy to see that for some constant $C > 0$ (independent of ε_*), and $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon_*$,

$$|u_{\bar{\lambda}}^{-p}(\eta) v_{\bar{\lambda}}^{-q}(\eta) - u_\lambda^{-p}(\eta) v_\lambda^{-q}(\eta)| \leq C\varepsilon_*, \ \lambda \leq |\eta| \leq \bar{\lambda} + 1.$$

Using the mean value theorem, we have, for $\lambda \leq |\xi| \leq \bar{\lambda} + 1$, that

$$\begin{aligned}
\int_{\Sigma_\lambda \setminus \Sigma_{\bar{\lambda}+1}} K(0, \lambda; \xi, \eta) \, d\eta &= \int_{\Sigma_\lambda \setminus \Sigma_{\bar{\lambda}+1}} \big(\big(\frac{|\xi|}{\lambda}\big)^\tau |\xi^\lambda - \eta|^\tau - |\xi - \eta|^\tau\big) d\eta \\
&= \int_{\Sigma_\lambda \setminus \Sigma_{\bar{\lambda}+1}} \big[\big(\big(\frac{|\xi|}{\lambda}\big)^\tau - 1\big) |\xi^\lambda - \eta|^\tau + \big(|\xi^\lambda - \eta|^\tau - |\xi - \eta|^\tau\big)\big] d\eta \\
&\leq C(|\xi| - \lambda) + \int_{\Sigma_\lambda \setminus \Sigma_{\bar{\lambda}+1}} \big(|\xi^\lambda - \eta|^\tau - |\xi - \eta|^\tau\big) d\eta \\
&\leq C(|\xi| - \lambda) + C|\xi^\lambda - \xi| \leq C(|\xi| - \lambda).
\end{aligned}$$

Thus, for $\varepsilon \in (0, \varepsilon_*)$, $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon$, $\lambda \leq |\xi| \leq \bar{\lambda} + 1$, from (13) it follows

$$\begin{aligned} u_\lambda(\xi) - u(\xi) &\geq -C\varepsilon \int_{\Sigma_\lambda \setminus \Sigma_{\bar{\lambda}+1}} K(0, \lambda; \xi, \eta) d\eta + \delta_1 \delta_2 (|\xi| - \lambda) \int_{\Sigma_{\bar{\lambda}+2} \setminus \Sigma_{\bar{\lambda}+3}} d\eta \\ &\geq \big(\delta_1 \delta_2 \int_{\Sigma_{\bar{\lambda}+2} \setminus \Sigma_{\bar{\lambda}+3}} d\eta - C\varepsilon\big)(|\xi| - \lambda) \geq 0. \end{aligned}$$

Along the same way, we can show

$$v_\lambda(\xi) - v(\xi) \geq 0 \ \text{ for } \ \bar{\lambda} \leq \lambda \leq \bar{\lambda} + \varepsilon, \lambda \leq |\xi| \leq \bar{\lambda} + 1.$$

Step 2 is established, and this completes the proof of Lemma 3.3. □

The following two key calculus lemmas are needed to carry out the final steps of the proof of Theorem 1.2.

Lemma 3.4. *(Lemma 5.7 in [18]) For $n \geq 1, \mu \in \mathbb{R}$, let f be a function defined on $\mathbb{R}^n$ and valued in $[-\infty, +\infty]$ such that*

$$\big(\frac{\lambda}{|y - x|}\big)^\mu f\big(x + \frac{\lambda^2 (y - x)}{|y - x|^2}\big) \geq f(y), \quad \forall\, |y - x| > \lambda > 0,\ x,\ y \in \mathbb{R}^n.$$

Then $f \equiv$ constant or $\pm\infty$.

Lemma 3.5. *(Lemma 5.8 in [18]) For $n \geq 1, \mu \in \mathbb{R}$, let $f \in C^0(\mathbb{R}^n)$, and $\mu \in \mathbb{R}$. Suppose that for every $x \in \mathbb{R}^n$, there exists $\lambda = \lambda(x) \in \mathbb{R}$ such that*

$$\big(\frac{\lambda}{|y - x|}\big)^\mu f\big(x + \frac{\lambda^2 (y - x)}{|y - x|^2}\big) = f(y), \quad \forall y \in \mathbb{R}^n \setminus \{x\}.$$

Then there are $a \geq 0, d > 0$ and $\bar{x} \in \mathbb{R}^n$, such that

$$f(x) \equiv \pm a\big(\frac{1}{d + |x - \bar{x}|^2}\big)^{\mu/2}.$$

Proof of Theorem 1.2. First, we show that there exists some $x_0 \in \mathbb{R}^n$ such that $\bar{\lambda}(x_0) < \infty$. Then we show that this implies $\bar{\lambda}(x)$ is finite for all $x \in \mathbb{R}^n$. We prove the former statement by contradiction. That is, assume otherwise, i.e., if $\bar{\lambda}(x) = \infty$ for all $x \in \mathbb{R}^n$, then for $\xi \in \mathbb{R}^n$,

$$u_{x,\lambda}(\xi) \geq u(\xi), \ \text{and}\ v_{x,\lambda}(\xi) \geq v(\xi), \quad \forall\, |\xi - x| > \lambda.$$

By Lemma 3.4, we conclude that $u = v = constant$, which cannot satisfy (1).

Now, for a fixed $x \in \mathbb{R}^n$, it follows from the definition of $\bar{\lambda}(x)$ that,

$$u_{x,\lambda}(\xi) \geq u(\xi),\ \forall\, 0 < \lambda < \bar{\lambda}(x),\ \forall\, |\xi - x| \geq \lambda.$$

From Lemma 1.3 (ii), we have, for any $\lambda \in (0, \bar{\lambda}(x))$, that

$$0 \quad < \quad a = \lim_{|\xi| \to \infty} |\xi|^{-\tau} u(\xi) \leq \lim_{|\xi| \to \infty} |\xi|^{-\tau} u_{x,\lambda}(\xi) = \lambda^{-\tau} u(x).$$

This shows $\bar{\lambda}(x) < \infty$ for all $x \in \mathbb{R}^n$. From Lemma 3.3, we have

$$u_{x,\bar{\lambda}}(\xi) = u(\xi),\ \text{and}\ v_{x,\bar{\lambda}}(\xi) = v(\xi),\ \forall\, x, \xi \in \mathbb{R}^n.$$

Invoking Lemma 3.5, we get

$$u(\xi) = c_1 \big(|\xi - \xi_0|^2 + d\big)^{\tau/2}$$

and

$$v(\xi) = c_2 \big(|\xi - \xi_0|^2 + d\big)^{\tau/2}$$

for some c_1, $c_2 > 0, d > 0$ and $\xi_0 \in \mathbb{R}^n$. □

4. **Non-existence of positive solutions in the subcritical case.** In this section, we give the proof of Theorem 1.5. To this end, we need the following lemmas.

Lemma 4.1. *Assume the same conditions as those in Theorem 1.5. For each $x \in \mathbb{R}^n$ there exists $\lambda_0(x) > 0$ such that*

$$u_{x,\lambda}(\xi) \geq u(\xi) \quad \textit{and} \quad v_{x,\lambda}(\xi) \geq v(\xi), \qquad \forall\, \xi \in \Sigma_{x,\lambda}, \forall\, 0 < \lambda < \lambda_0(x).$$

The proof is the same as that of Lemma 3.2, so we omit the details.

Lemma 4.2. $\bar{\lambda}(x) = \infty$ *for all* $x \in \mathbb{R}^n$.

Proof. We prove it by contradiction. Assume the contrary, that is, there exists some $x_0 \in \mathbb{R}^n$ such that $\bar{\lambda}(x_0) < \infty$. By the definition of $\bar{\lambda}$,

$$\begin{aligned} u_{x_0,\bar{\lambda}}(\xi) &\geq u(\xi), \\ v_{x_0,\bar{\lambda}}(\xi) &\geq v(\xi), \end{aligned}$$

for $\xi \in \Sigma_{x_0,\bar{\lambda}}$. From (10) and (11) with $x = x_0, \lambda = \bar{\lambda}$ and the fact that at least one of the parameters θ_1 and θ_2 is positive, we have

$$\begin{aligned} u_{x_0,\bar{\lambda}}(\xi) &> u(\xi), \\ v_{x_0,\bar{\lambda}}(\xi) &> v(\xi), \end{aligned}$$

for $\xi \in \Sigma_{x_0,\bar{\lambda}}$. Similar to the arguments in the proof of Lemma 3.3, we can conclude that there is a suitably small $\varepsilon > 0$ such that for $\lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon)$

$$\begin{aligned} u_{x_0,\lambda}(\xi) &\geq u(\xi), \\ v_{x_0,\lambda}(\xi) &\geq v(\xi), \end{aligned}$$

for $\xi \in \Sigma_{x_0,\lambda}, \lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon)$. This contradicts with the definition of $\bar{\lambda}$, and this completes the proof. ☐

Proof of Theorem 1.5. According to Lemma 4.2, $\bar{\lambda}(x) = \infty$ for all $x \in \mathbb{R}^n$, that is, for all $\lambda > 0$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} u_{x,\lambda}(\xi) &\geq u(\xi) \quad \text{for } \xi \in \Sigma_{x,\lambda}, \\ v_{x,\lambda}(\xi) &\geq v(\xi) \quad \text{for } \xi \in \Sigma_{x,\lambda}. \end{aligned}$$

By Lemma 3.4 we conclude that $u = v = \textit{constant}$, but this cannot satisfy system (1). Thus, we arrive at a contradiction, and this completes the proof. ☐

Acknowledgments. The authors would like to thank the referees for carefully reading this manuscript and providing valuable comments and suggestions.

REFERENCES

[1] G. Caristi, L. D'Ambrosio and E. Mitidieri, Representation formulae for solutions to some classes of higher order systems and related Liouville theorems, *Milan J. Math.*, **76** (2008), 27–67.

[2] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, *Comm. Pure Appl. Math.*, **59** (2006), 330–343.

[3] W. Chen, C. Li and B. Ou, Classification of solutions for a system of integral equations, *Comm. Partial Differential Equations*, **30** (2005), 59–65.

[4] W. Chen, C. Li and B. Ou, Qualitative properties of solutions for an integral equation, *Discrete Contin. Dyn. Syst*, **12** (2005), 347–354.

[5] W. Chen and C. Li, An integral system and the Lane-Emdem conjecture, *Discrete Contin. Dyn. Syst.*, **24** (2009), 1167–1184.

[6] J. Dou, Liouville type theorems for the system of integral equations, *Appl. Math. Comput.*, **217** (2010), 2586–2594.
[7] J. Dou and M. Zhu, Sharp Hardy-Littlewood-Sobolev inequality on the upper half space, *Int. Math. Res. Not.*, **3** (2015), 651–687.
[8] J. Dou and M. Zhu, Reversed Hardy-Littlewood-Sobolev inequality, *Int. Math. Res. Not.*, **19** (2015), 9696–9726.
[9] M. Ghergu, Lane-Emden systems with negative exponents, *J. Funct. Anal.*, **258** (2010), 3295–3318.
[10] Y. Han and M. Zhu, Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds and applications. *J. Differential Equations*, **260** (2016), 1–25.
[11] F. Hang, On the integral systems related to Hardy-Littlewood-Sobolev inequality, *Math. Res. Lett.*, **14** (2007), 373–383.
[12] Y. Lei, On the integral systems with negative exponents, *Discrete Contin. Dyn. Syst.*, **35** (2015), 1039–1057.
[13] Y. Lei and C. Li, Sharp criteria of Liouville type for some nonlinear systems, *Discrete Contin. Dyn. Syst.*, **36** (2016), 3277–3315.
[14] C. Li and J. Villavert, A degree theory framework for semilinear elliptic systems, *Proc. Amer. Math. Soc.*, **144** (2016), 3731–3740.
[15] C. Li and J. Villavert, Existence of positive solutions to semilinear elliptic systems with supercritical growth, *Comm. Partial Differential Equations*, **41** (2016), 1029–1039.
[16] Y. Y. Li and M. Zhu, Uniqueness theorems through the method of moving spheres, *Duke Math. J.*, **80** (1995), 383–417.
[17] Y. Y. Li and L. Zhang, Liouville type theorems and Harnack type inequalities for semilinear elliptic equations, *J. Anal. Math.*, **90** (2003), 27–87.
[18] Y. Y. Li, Remark on some conformally invariant integral equations: The method of moving spheres, *J. Eur. Math. Soc. (JEMS)*, **6** (2004), 153–180.
[19] E. H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.*, **118** (1983), 349–374.
[20] J. Liu, Y. Guo and Y. Zhang, Existence of positive entire solutions for polyharmonic equations and systems, *J. Partial Differential Equations*, **19** (2006), 256–270.
[21] L. Ma and D. Chen, A Liouville type theorem for an integral system, *Commun. Pure Appl. Anal.*, **5** (2006), 855–859.
[22] Q. A. Ngô and V. H. Nguyen, Sharp Reversed Hardy-Littlewood-Sobolev inequality on $\mathbf{R}^n$, *Israel J. Math.*, To Appear.
[23] J. Wei and X. Xu, Classification of solutions of higher order conformally invariant equations, *Math. Ann.*, **313** (1999), 207–228.
[24] X. Xu, Uniqueness theorem for integral equations and its application, *J. Funct. Anal.*, **247** (2007), 95–109.
[25] Z. Zhang, Positive solutions of Lane-Emden systems with negative exponents: Existence, boundary behavior and uniqueness, *Nonlinear Anal.*, **74** (2011), 5544–5553.

E-mail address: jbdou@xaufe.edu.cn
E-mail address: renff93@126.com
E-mail address: john.villavert@utrgv.edu, john.villavert@gmail.com